The properties of the paper size $1:\sqrt{2}$ (*)

Teaching suggestions for a mathematical laboratory with paper folding

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(*) A. Criscuolo, Le proprietà del formato della carta 1 : √2: spunti didattici per un laboratorio matematico con la piegatura della carta, L'insegnamento della matematica e delle scienze integrate, vol. 38b n. 1 - febbraio 2015

Didactics and Research of Folding
10th International Conference
Freiburg im Breisgau, Germany, 13-15 November 2015

Before starting a premise to be shared

This workshop is not intended as an activity to learn how to build a model

The proposals of the workshops for the students are:

- to be able to play their skills with geometric concepts;
- to discover / to rediscover geometric and arithmetic properties embedded in a folded paper;
- to experience and to discover unexpected geometric properties.

So, I think that the best way to present the workshop is creating here the same atmosphere as in the class.

Therefore I ask you to perform the role of the young students while I play the role of the old teacher.

Introduction

The workshop offers a paper folding activity that leads to the discovery of mathematical properties "incorporated" into the A4 sheet.

The mathematical properties of the paper format 1:√2, the common A4 sheet, are well known and widely used by lovers of origami, but **they are less known and especially poorly used in mathematics teaching.**

In this workshop we present several properties of the format 1:√2, some known and other unexpected: we will identify mathematical objects and concepts with activities related to the folding of the paper for the secondary school classes (students 12 -15 years old).

The paper format A0-A10 ISO 216 and its characteristics

The main characteristics of the sheet series A0-A10.

- Every sheet could be obtained by halving the previous: there is no wasted paper
- The sheet has same shape: with an aesthetic and a practical advantage

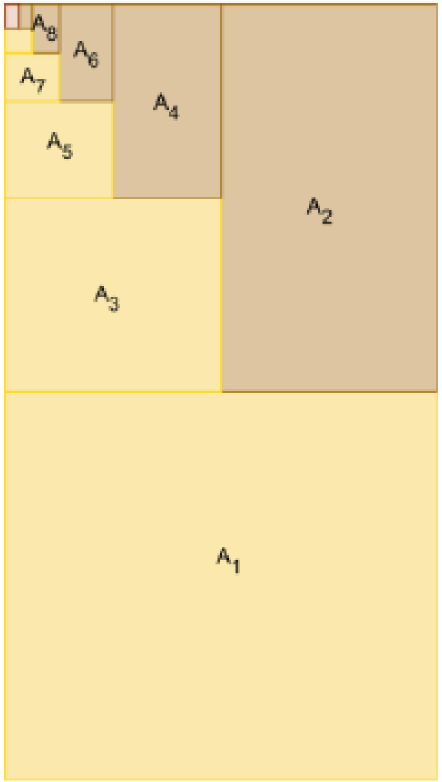
As you know these advantages depend on the particular relationship between the size of the paper: $1 : \sqrt{2}$

This was noted in 1786 by the German scientist Georg Christoph Lichtenberg.

The format was developed later in France, during the French Revolution, and introduced as a DIN standard (DIN 476) in Germany in 1922.

Today the standard paper sizes ISO 216, like the A4, are widely used all over the world.

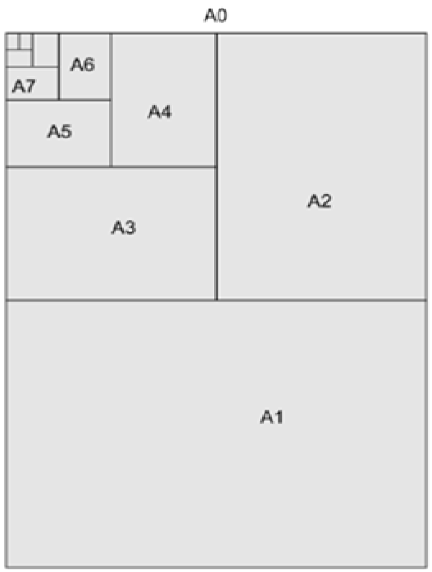
We begin by folding any kind of rectangular sheet

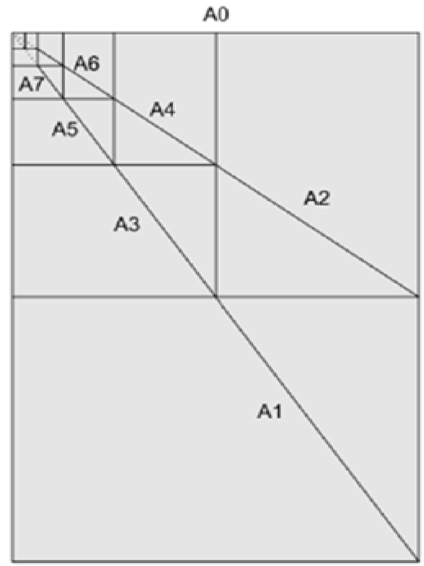


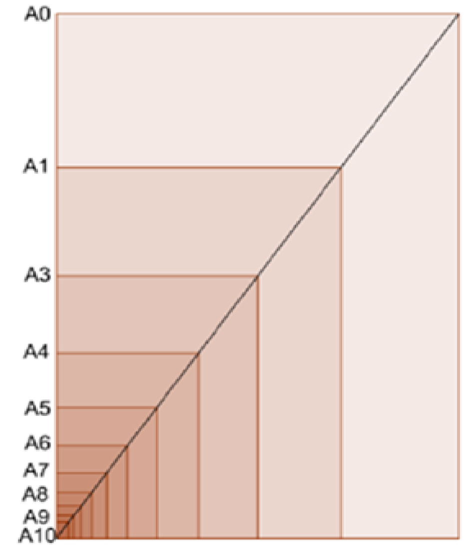
1 x 1,8

Is interesting to note that starting from whatever rectangular sheet and proceeding by successive halving, there will be realized two separate successions of similar rectangles between them.

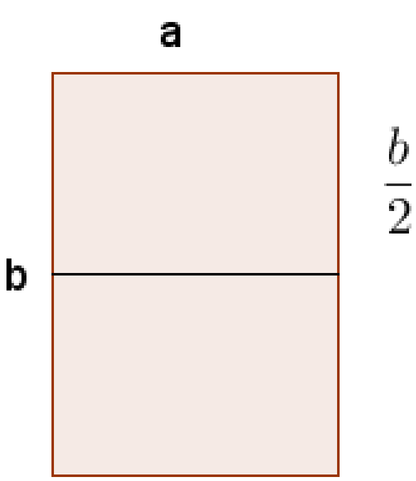
Folding an A4 sheet







Proceeding by successive halving of A4 paper we obtain a single set of similar rectangles



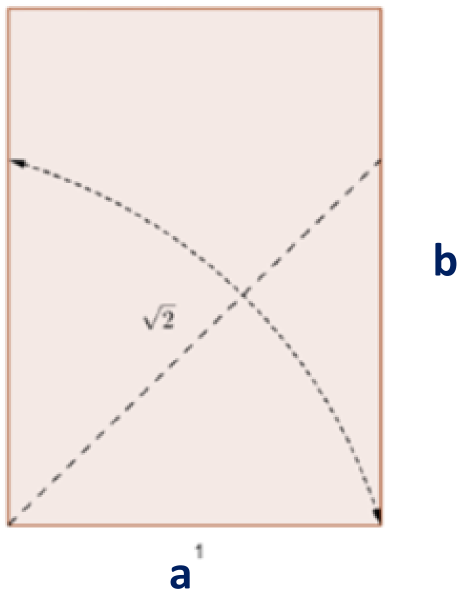
In fact to get similar rectangles must be $\frac{b}{a} = \frac{a}{\frac{b}{2}}$

from which it appears

$$b = \sqrt{2}a.$$

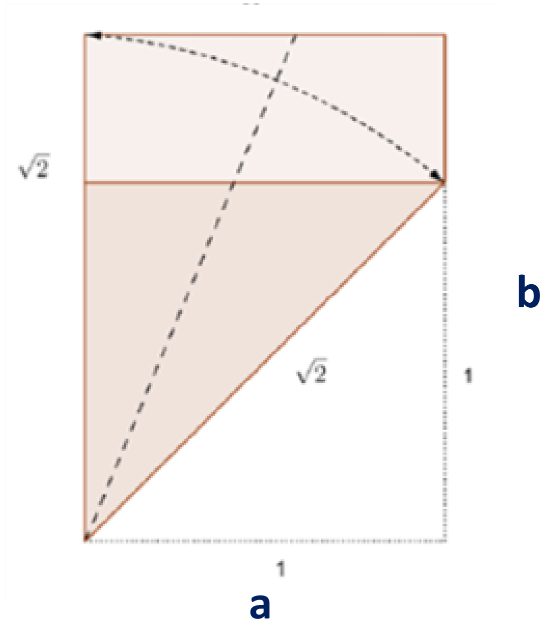
With two simple folds of a A4 sheet you can the discovery of the relationship $1:\sqrt{2}$

1° Step



$$\frac{b}{a} = \sqrt{2}$$

2° Step



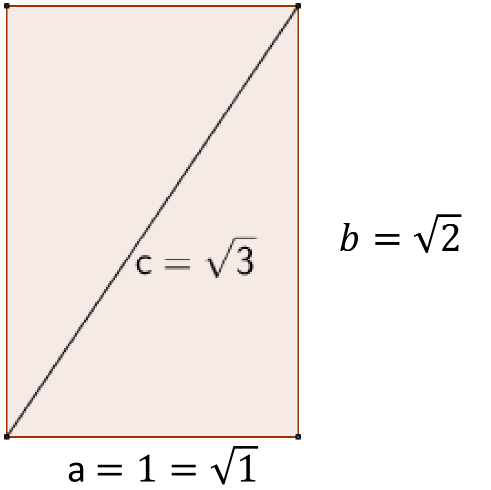
An interesting set of irrational numbers:
three irrational numbers embedded in a A4 sheet

$\sqrt{1}$ $\sqrt{2}$ $\sqrt{3}$ $1 + 2 = 3$

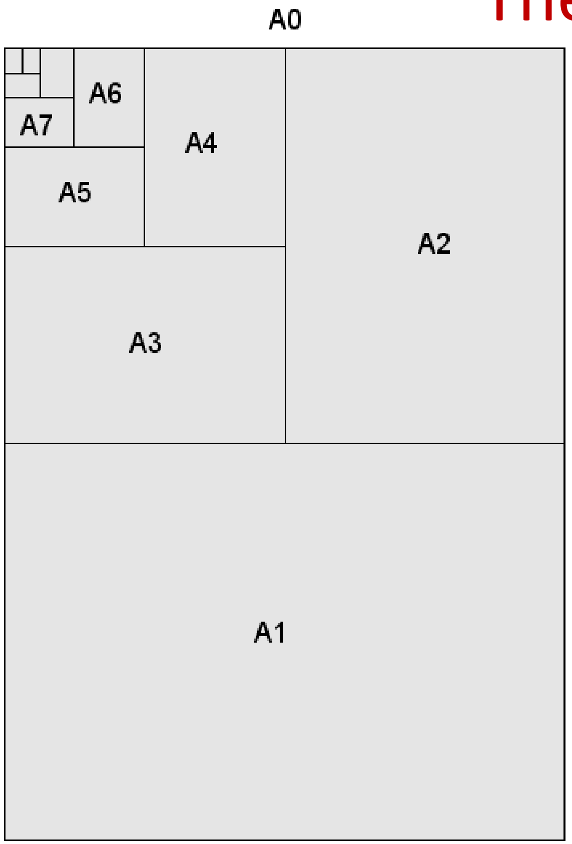
as a Pythagorean triple of natural numbers:

3 4 5

$9 + 16 = 25$



The arithmetics of the A4 sheet



Area A0 sheet = $1\ m^2$

$$Area\ A4 = \frac{1}{2^4} Area\ A0 = \frac{1}{16}\ Area\ A0$$

$$\frac{297}{210} = \frac{99}{70} = \mathbf{1,414}2857142857142857 \cong \sqrt{2}$$

Sheet size

	a(mm)	b(mm)
A0	841	1189
A1	594	841
A2	420	594
A3	297	420
A4	**210**	**297**
A5	148	210
A6	105	148
A7	74	105
A8	52	74
A9	37	52
A10	26	37

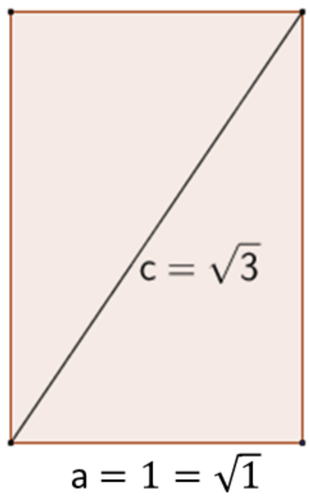
A rational number as an infinite periodic decimal $\frac{99}{70} = \mathbf{1,414}2857142857142857$

An-irrational number as an infinite aperiodic decimal $\sqrt{2} = \mathbf{1,414}2135623730950488016887$

$\sqrt{2} \cong \frac{99}{70}$ The difference is less than $0{,}005\ \%$

The arithmetics of the A4 sheet

Some issues that may emerge in the classroom



$b = \sqrt{2}$ $\dfrac{b}{a}$

A rational number as an infinite periodic decimal:

$$\frac{99}{70} = \mathbf{1,414}2857142857142857$$

An irrational number as an infinite periodic decimal"

$$\sqrt{2} = \mathbf{1,414}21356237309504880168872 \ldots$$

	a(mm)	b(mm)
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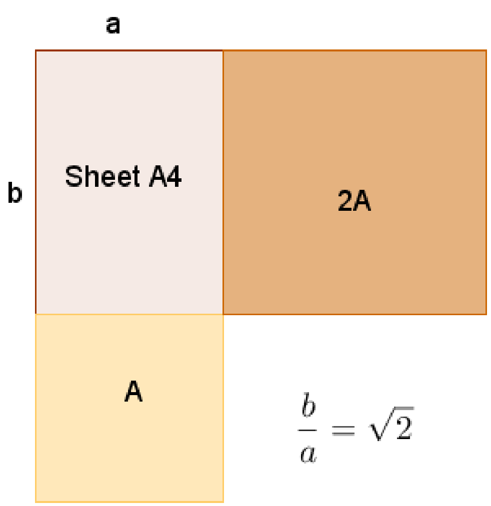
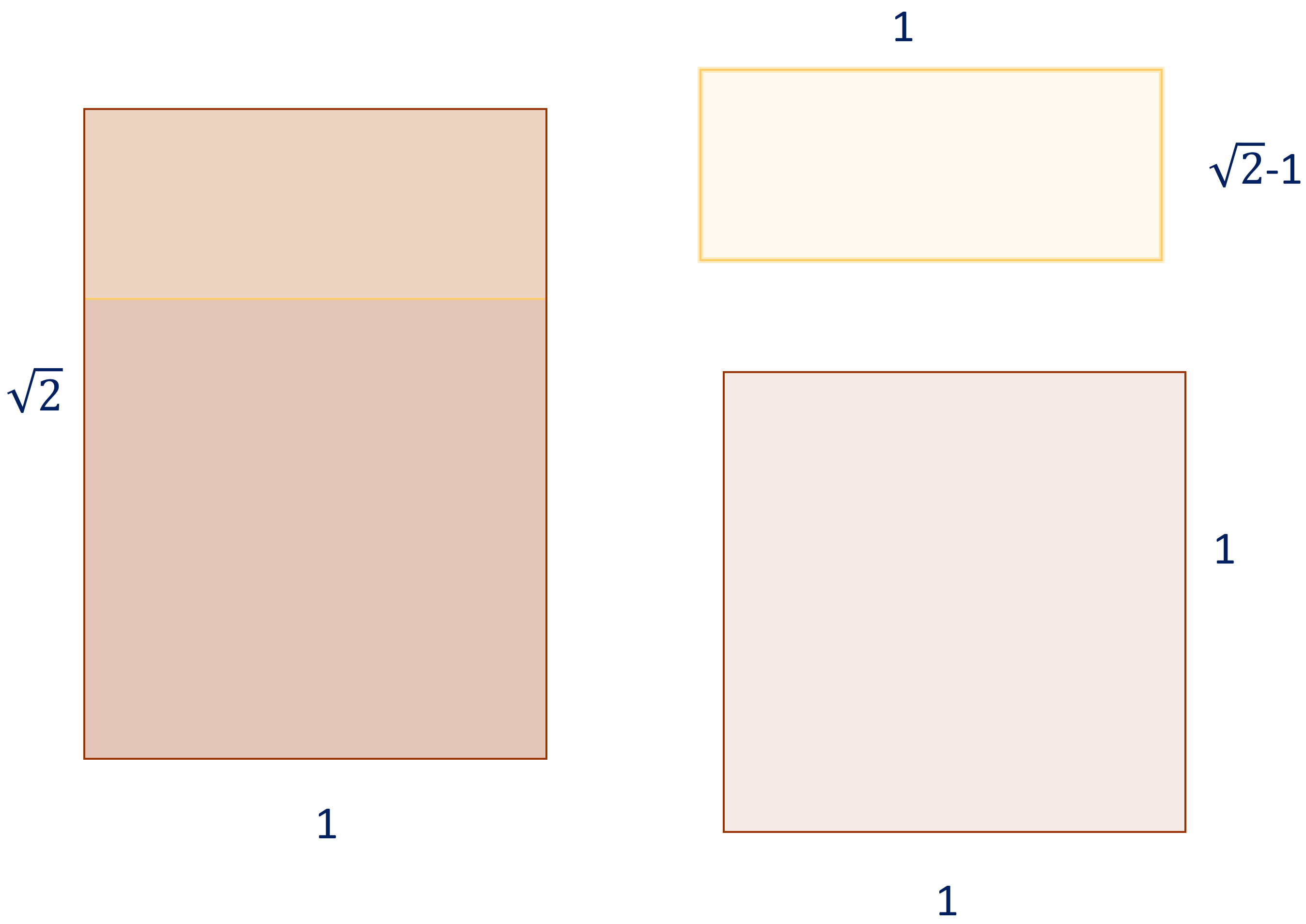
The pairs of natural numbers (a,b) are just approximated solutions of the equation $2a^2 - b^2 = 0$

Geometrical meaning of the equation $2a^2 - b^2 = 0$

The square built on the long side of the sheet is equivalent to the doble of the square built on the short side.

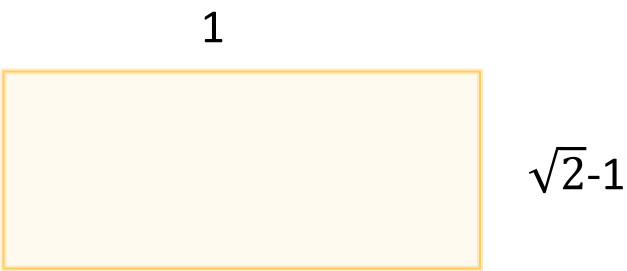


Cutting an A4 sheet

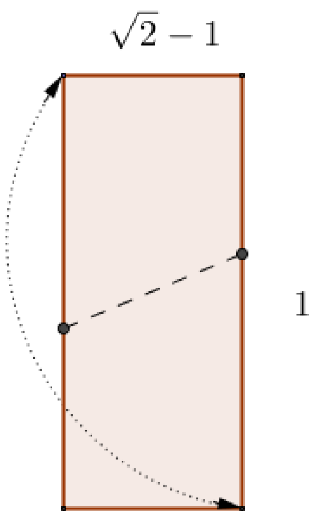


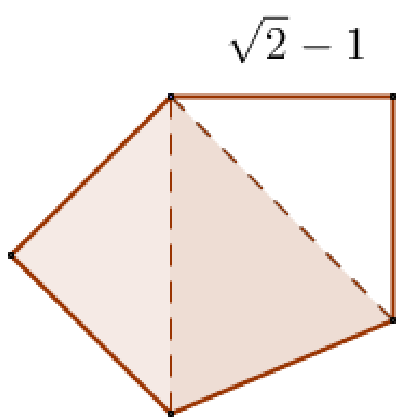
A4 leftover

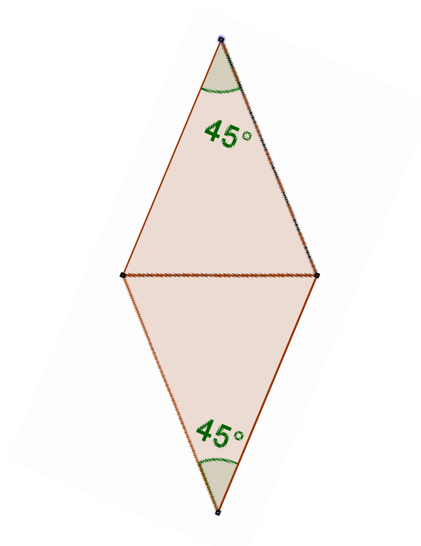
Geometric properties of the sheet A4, a particular strip: “A4 leftover”

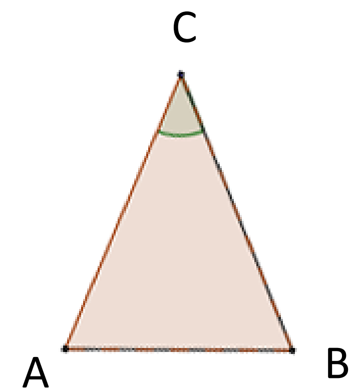


When from an A4 sheet you carve out a square, witch has the side equal to the short A4 side, you get a rectangular strip 1 × (√2-1) that shows an interesting property: it can be folded as a rhombus of angles 45 ° - 135 ° and subsequently an isosceles triangle with the vertex angle of 45 °.

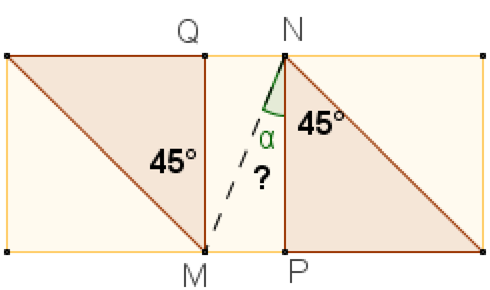








Some issues that may emerge in the classroom

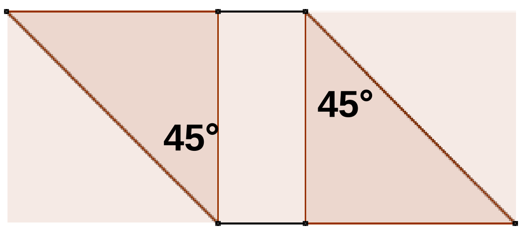


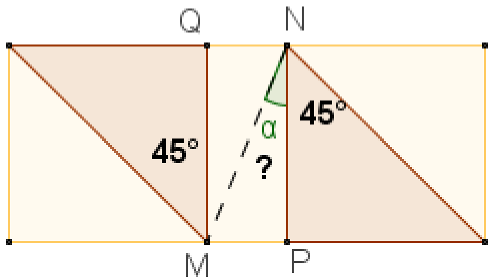
What’s the area of the isosceles triangle named ABC?

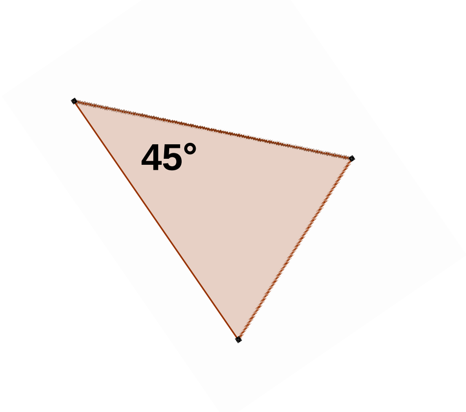
What is the measure of the MNP angle?

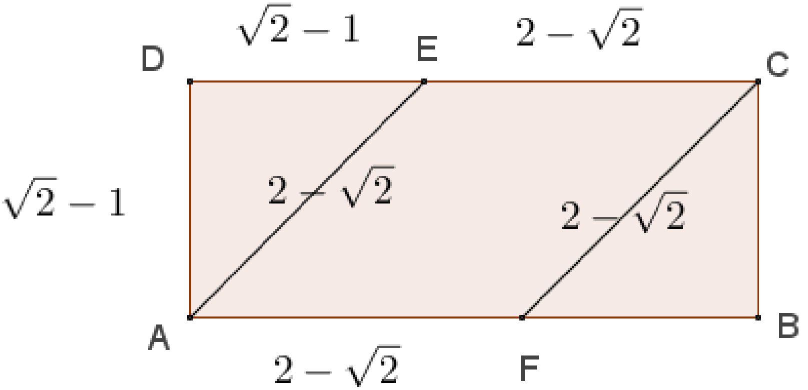
What’s the area of the MNP triangle?

Geometric properties of the sheet A4: a particular strip of A4 sheet



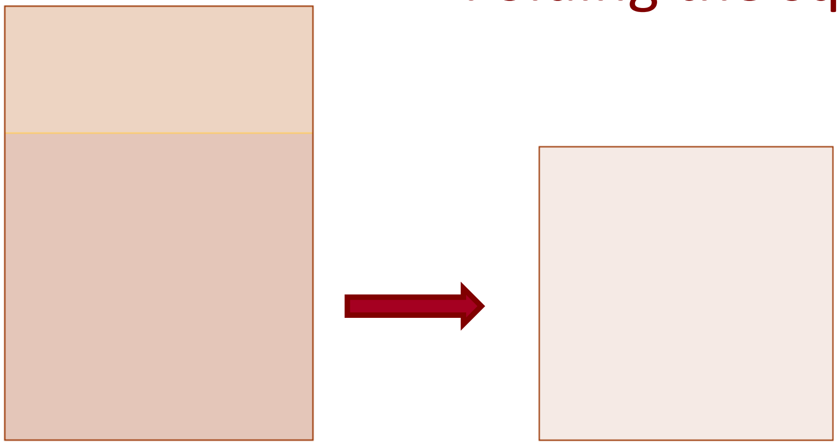




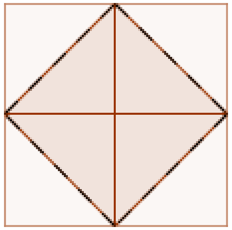


$$\overline{AF}=\overline{CE}=\overline{AB}-\overline{FB}=1-(\sqrt{2}-1)=2-\sqrt{2}$$

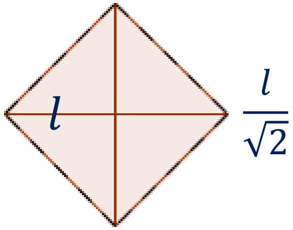
Folding the square part of A4 sheet



How to fold the squared sheet to obtain another square with the halved area

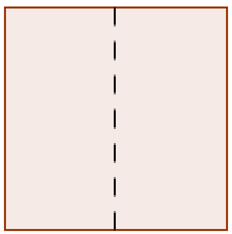


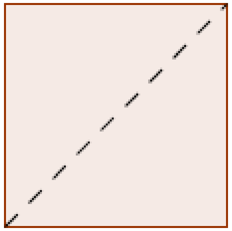
$$A = l^2$$

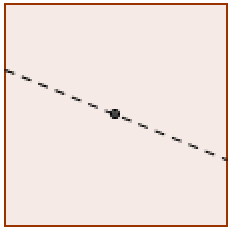


$$A = \frac{l^2}{2}$$

How to fold the squared sheet to obtain a polygon with an halved area







Medial axis

Diagonal axis

Center of symmetry

Halving a square

In Plato's dialog Meno

Folding an A4 sheet into 64 equal rectangles

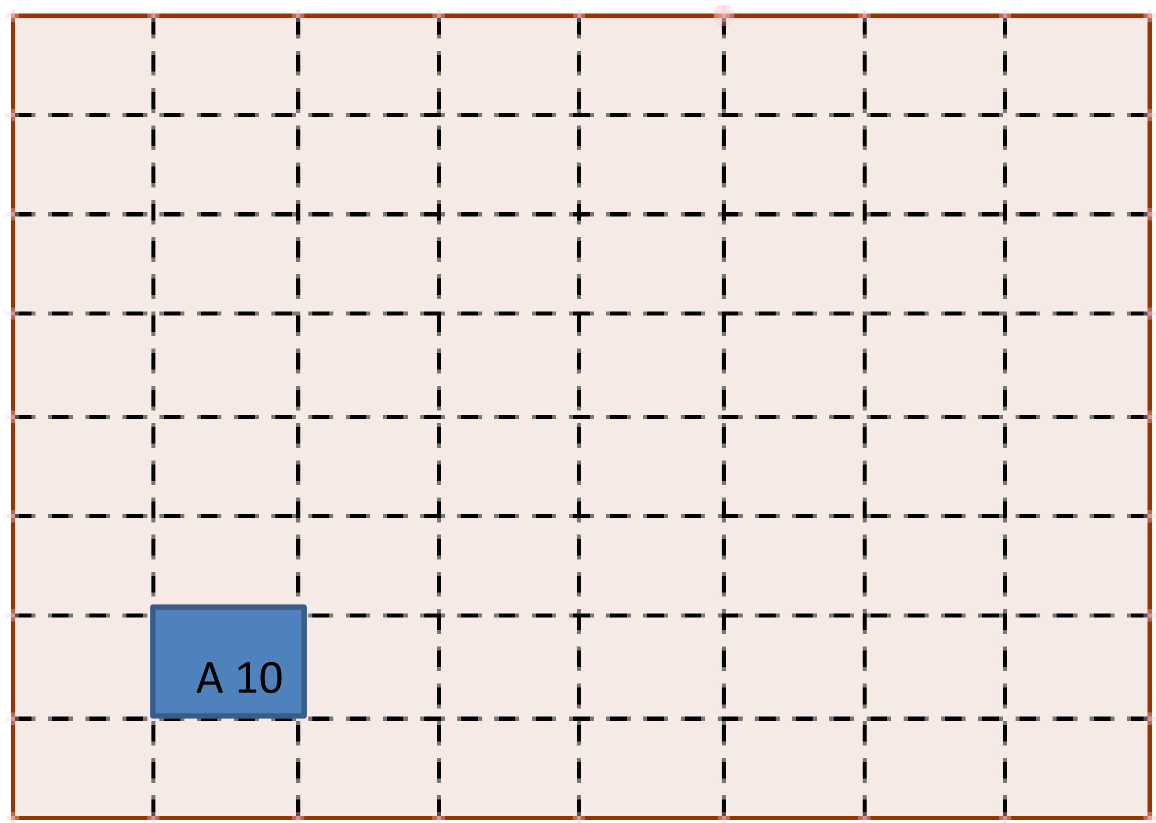
Some issues that may emerge in the classroom

The 64 rectangles are equal and they are similar to an element of A0-A10 series.

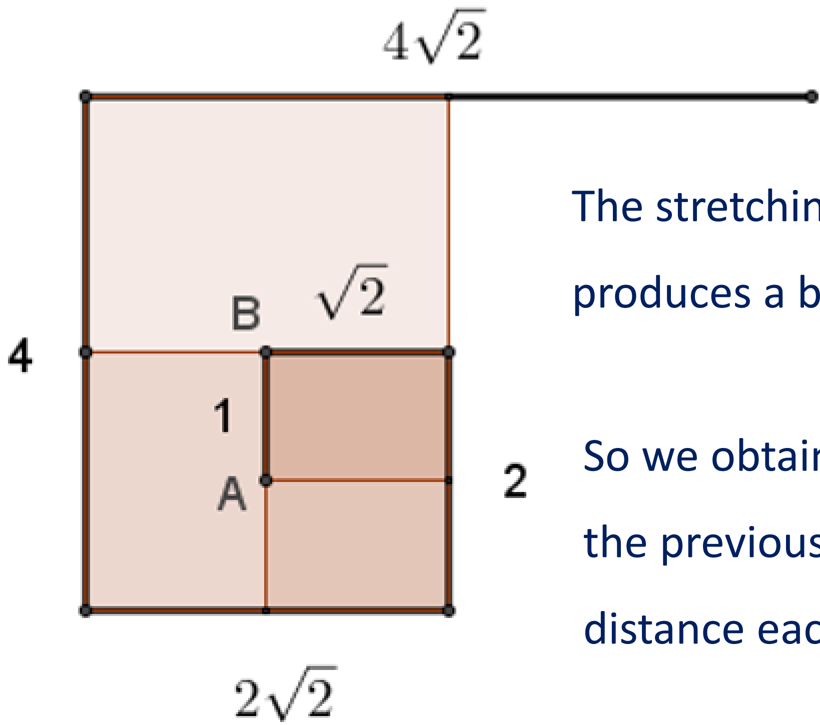
The rectangles that we got are equal to an element of the series. What is this element?

A 10

The series of rectangles A0 - A10 can be built doubling rectangles instead of halving.



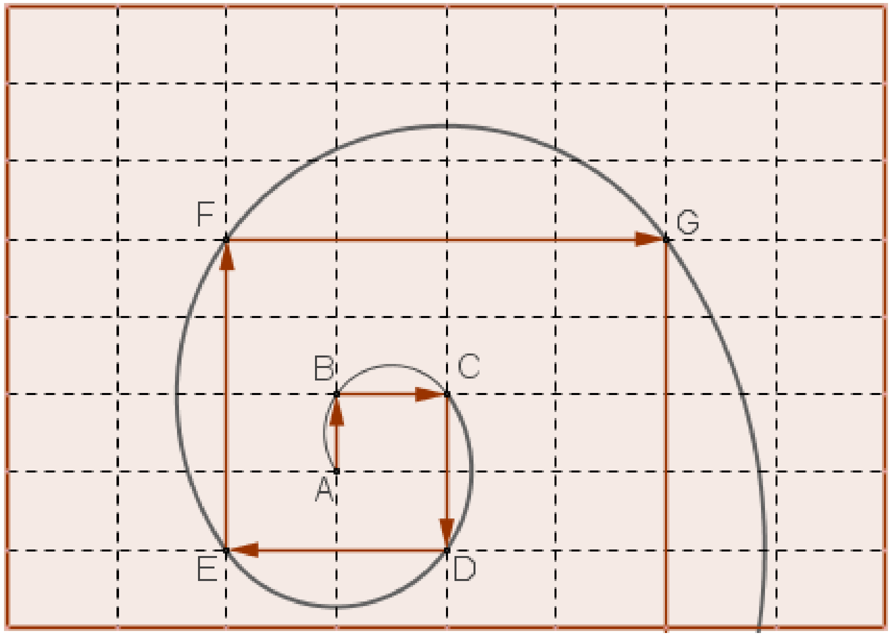
A path of sheets: A10 – A9 – A8 – A7 – A6 -……….



The stretching by a factor of 2, in the direction of the short side, produces a broken line formed of perpendicular segments.

So we obtain a spiral path of steps : each step is obtained from the previous, proceeding in a perpendicular direction with a distance each time increased by a factor of √2.

From a spiral of segments to a spiral of arches



For the sequence of vertices of the rectangles of the A0-A10 it is possible to pass a spiral of semi-circumference. **How to do?**

Why semi-circumference?

Three consecutive vertices (ABC, EDC, ...) are right-angled triangle vertices, the arches ABC, CDE, are semi-circumference.

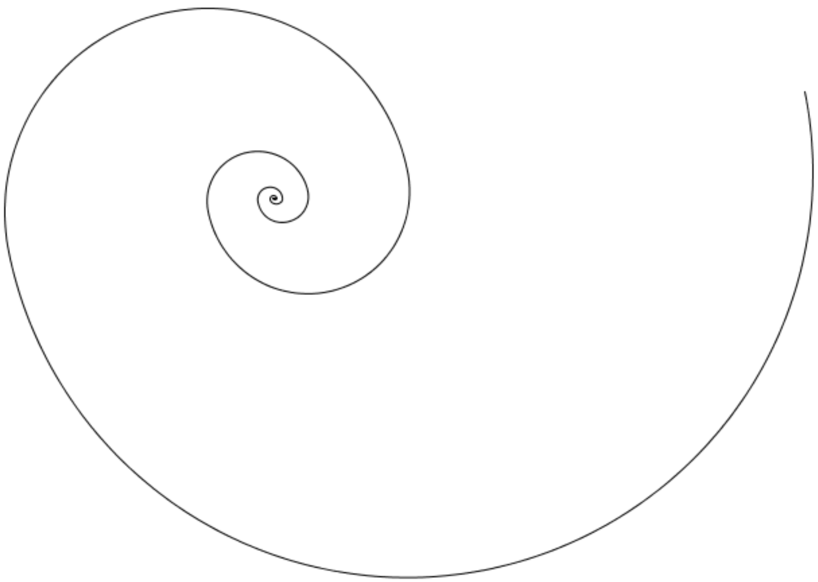
Geometric property: Right angle triangle inscript in a semicircle

Why the consecutive arches have a common tangent?

The points (C, E, G, ...), where the semi-circumference arches are connected, are aligned along the same straight line (the diagonal of the first rectangle) and are centers of the subsequent arc of the semi-circumference.

The tangents to the arcs are perpendicular to the radius

The equations of the rectangular spiral polycentric 1 : $\sqrt{2}$



If you assign three points in a Cartesian plane (the vertices of the triangle has its catheti 1 and √2) the coordinates of successive points that identify the spiral are determined by series:

$$x_n = x_{n-1} - 2(x_{n-2} - x_{n-3})$$
$$y_n = y_{n-1} - 2(y_{n-2} - y_{n-3});$$

Third order linear differential equation

Punto	x	y
A_0	0	0
A_1	0	1
A_2	$\sqrt{2}$	1

Thank you

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A. Criscuolo, Le proprietà del formato della carta 1 : $\sqrt{2}$*: spunti didattici per un laboratorio matematico con la piegatura della carta, L'insegnamento della matematica e delle scienze integrate, vol. 38b n. 1 - febbraio 2015*