SUBDIVISIONS INTO EQUAL PARTS VIA ORIGAMI: GEOMETRY AND BEYOND

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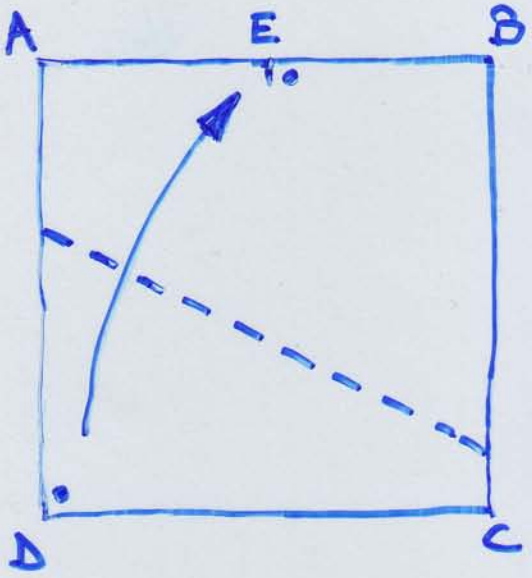
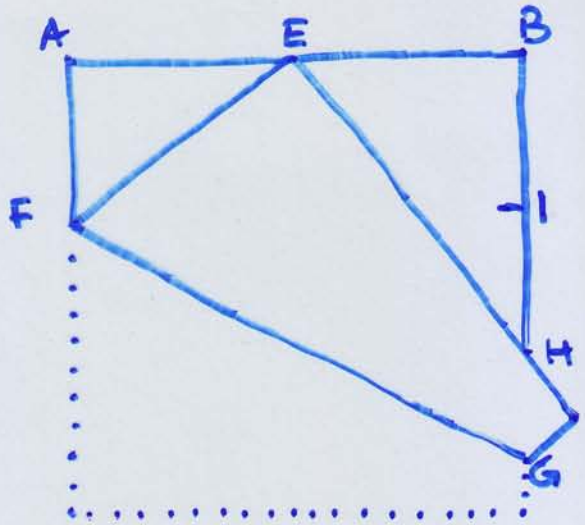
HAGA'S THEOREM





$AE = \frac{1}{2} AB \implies BI = \frac{1}{3} BC$

- $AB = 1 \quad AE = \frac{1}{2} \quad AF = x \quad FE = 1 - x$

$$x^2 + \left(\frac{1}{2}\right)^2 = (1-x)^2 \quad \dots \quad x = \frac{3}{8}$$

- AEF and EBH similar $\Rightarrow$

$AF : AE = EB : BH$

$EB = \frac{1}{2} \qquad BH = y$

$$\frac{3}{8} : \frac{1}{2} = \frac{1}{2} : y$$

$$y = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{8}{3} = \frac{2}{3}$$

$$BI = \frac{1}{3}$$

What if $AE = \frac{1}{m}$? $EB = \frac{m-1}{m}$ ... $BI = \frac{1}{m+1}$

VARIATION 1

$\frac{1}{n} \xrightarrow{(HT)} \frac{1}{n+1}$

From the first n-subdivision pt. on one side we obtain the first (n+1)-subdivision pt. on the next side.

How can we use all this efficiently?

Example $N = 13$

$\frac{1}{13} \leftarrow \frac{1}{12} \leftarrow \frac{1}{6} \leftarrow \frac{1}{3} \leftarrow \frac{1}{2} \leftarrow 1$

b : bisection of left part of top side

h : application of HT, i.e.

- fold bottom-left corner to current pt.
- rotate ↰ by 90°
- bisect left part of "new" top side
- open up the paper

The word $w_{13} = b\,h\,b\,b\,h$ encodes the instructions for $\frac{1}{13}$

In general, we have obtained an algorithm.

The denominators of above fractions are

13 , 12, 6, 3, 2, 1

Each is obtained from the previous one either by subtracting 1 (odd case) or by halving it (even case):

$$13 \xrightarrow[-1]{} 12 \xrightarrow[:2]{} 6 \xrightarrow[:2]{} 3 \xrightarrow[-1]{} 2 \xrightarrow[:2]{} 1$$

Working <u>backwards</u>, we get w_{13} just replacing ":2" with b and "-1" with h.

Examples

$12 \to 6 \to 3 \to 2 \to 1$

$w_{12} = b\,h\,b\,b$

$11 \to 10 \to 5 \to 4 \to 2 \to 1$

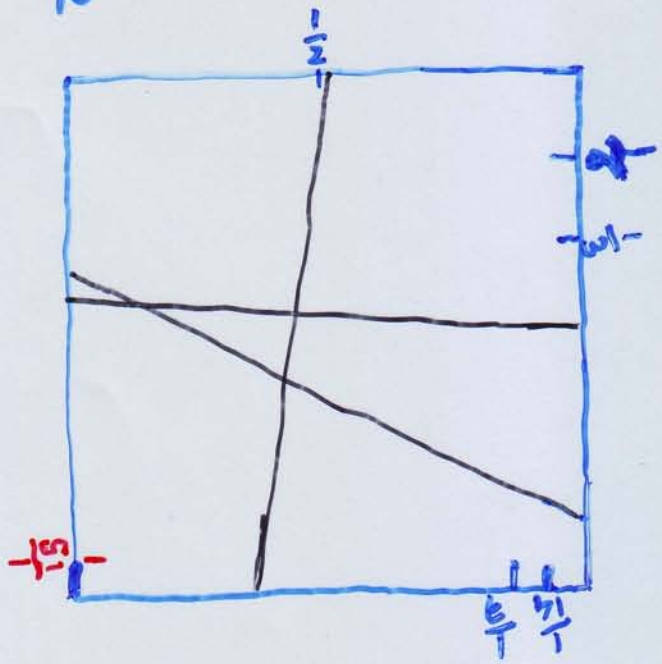
$w_{11} = b\,b\,h\,b\,h$

The algorithm is efficient and related to the base 2 representation of N.
(efficiency : cf. Horner algorithm.
base 2 : given n in base 2, determine w_n - homework).

VARIATION 2

Remark When folding $\frac{1}{N}$, the number of creases across the paper is the number of h's in w_N; For instance if $N = 15$

$w_{15} = bhbhbh.$



So far, we used HT in the form $\frac{1}{n} \to \frac{1}{n+1}$.

But in the same way we can show that

$$\frac{m}{n} \xrightarrow{HT} \frac{m}{n+m}$$

Can we use this generalized form just once (for suitable m, n) in order to get $\frac{1}{N}$?

Yes, and we can choose n to be a power of 2.

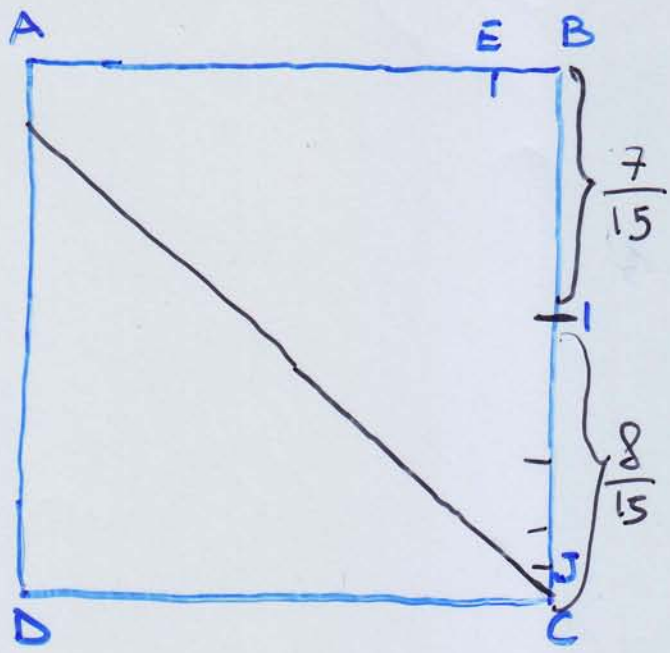
Example $N = 15$

$15 = 2^3 + 7$

$\frac{7}{8} \xrightarrow{HT} \frac{7}{15}$

$AE = \frac{7}{8} \Rightarrow BI = \frac{7}{15} \Rightarrow$

$IC = \frac{8}{15}$



By repeated bisection, get $CJ = \frac{1}{15}$

In general

- write $N = 2^k + m$, $0 \le m < 2^k$
- locate E on AB such that $AE = \frac{m}{2^k}$
- via HT, get I on BC such that

$$BI = \frac{m}{2^k + m} = \frac{m}{N}$$

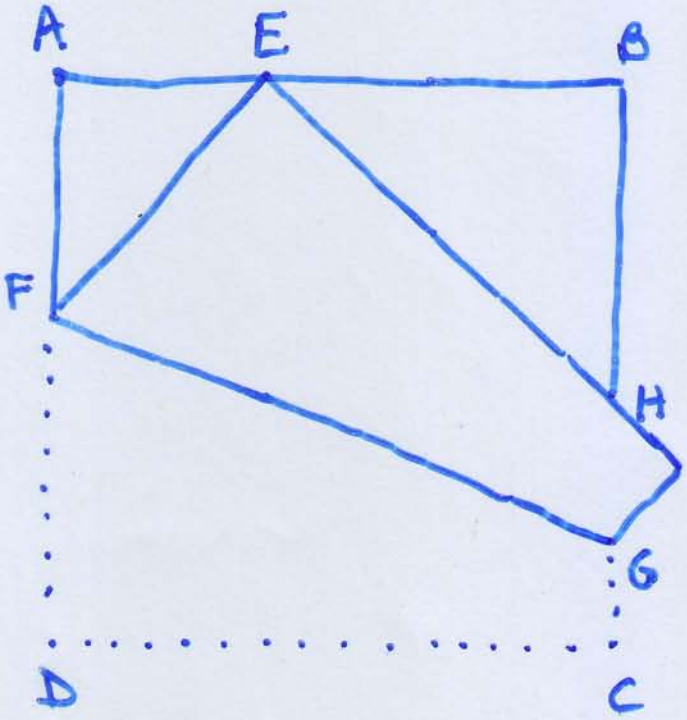
$$\Rightarrow IC = 1 - BI = \frac{2^k}{N}$$

- bisect IC k times and get $JC = \frac{1}{N}$

Comparison

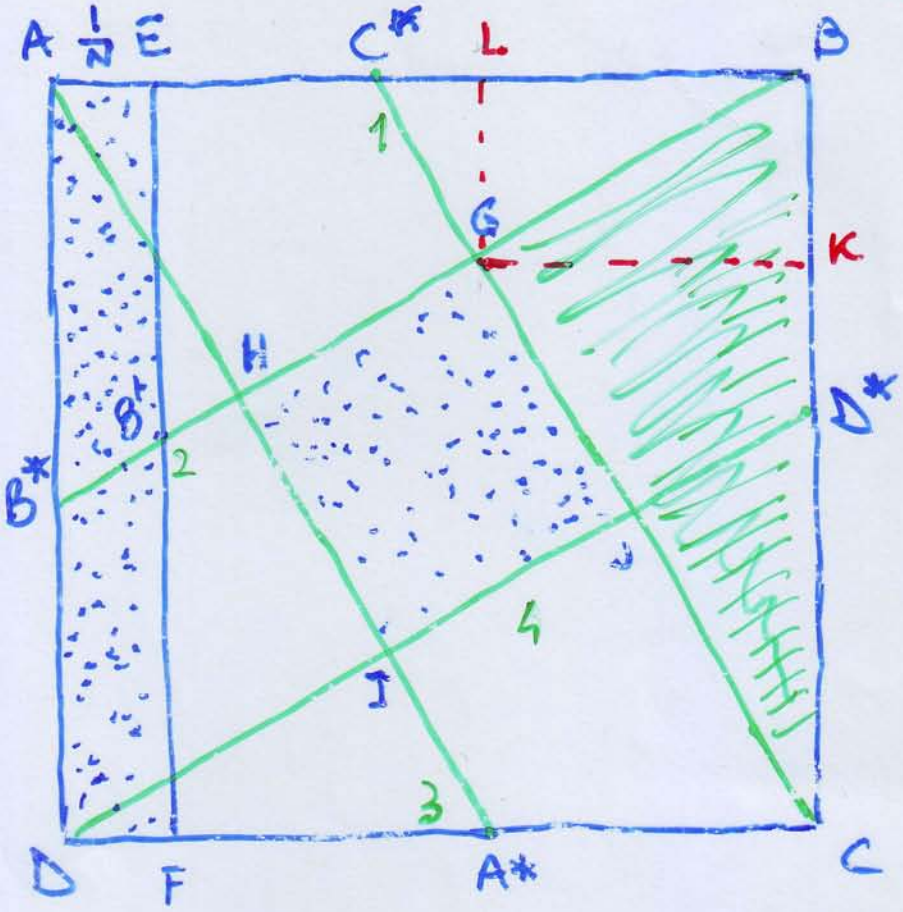
Theory both methods are exact

Practice folding problems when m is close to 2^k.



- $BH > AF$
- $E\hat{H}B$ small

From the strip to the square



Construct a square equivalent to rectangle ADFE.

Abe 2003

Fold 1 (B → EF), 2, 3, 4

G is midpoint of BB' ⇒

L is midpoint of EB ⇒

$$GK = LB = \frac{1}{2}\cdot\left(1-\frac{1}{N}\right) = \frac{N-1}{2N}$$

$$\text{Area}(BCG) = \frac{1}{2}\cdot 1\cdot\frac{N-1}{2N} = \frac{N-1}{4N}$$

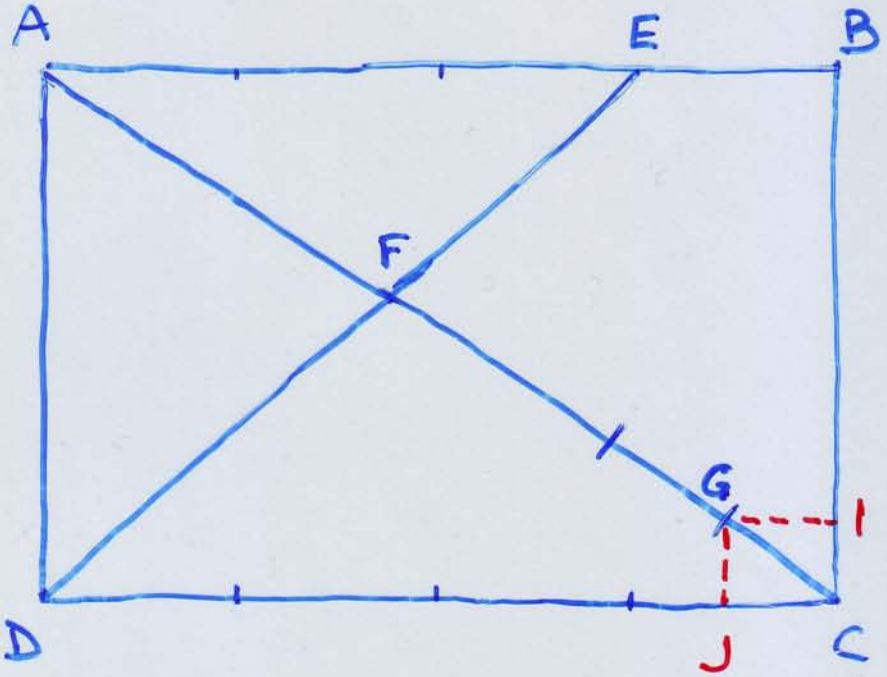
$$\text{Area}(GHIJ) = 1^2 - 4\cdot\frac{N-1}{4N} =$$

$$= 1-\frac{N-1}{N} = \frac{1}{N}$$

RECTANGLES A

Example

$N = 7 = 2^2 + 3$



E on AB s.t.

$\frac{AE}{AB} = \frac{3}{4}$

$F = AC \cap DE$

$\Rightarrow \frac{AF}{FC} = \frac{3}{4}$ (*) by triangle similarity

$\Rightarrow \frac{AF}{AC} = \frac{3}{7}$ and $\frac{FC}{AC} = \frac{4}{7}$

Bisect FC twice and get G s.t.

$\frac{GC}{AC} = \frac{1}{7}$

Project G and get I, J s.t.

$\frac{IC}{BC} = \frac{1}{7}$, $\frac{JC}{DC} = \frac{1}{7}$

(*) can also be proved via Lang's crossing-diagonals technique.

RECTANGLES B

Fujimoto's Approximation

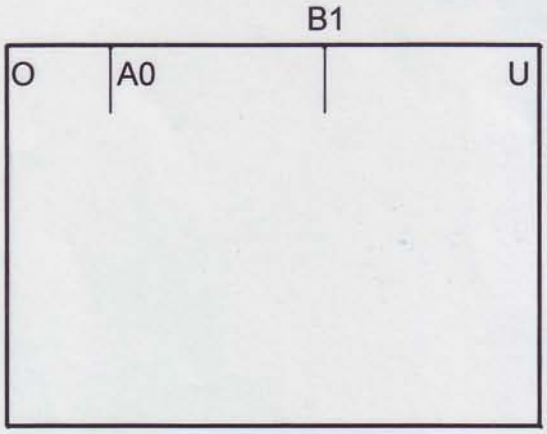
"Soft" introduction to the concept of limit (and of fixed point of a function)

It is rapidly convergent.

In real life, any construction is approximate, so an easy approximate construction is accurate than a complicated theoretically exact construction.

Moreover, in this construction we can control the approximation.

DIVISION IN 7 EQUAL PARTS

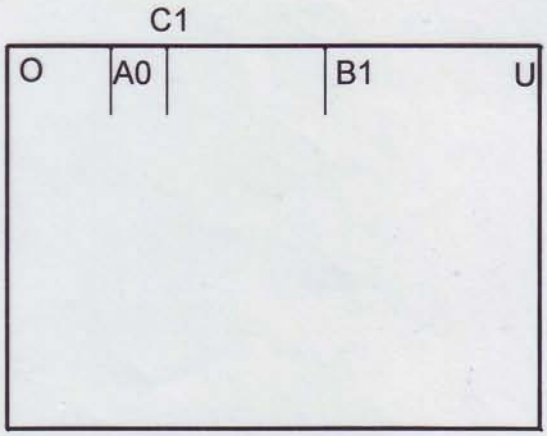


A_0 any pt on OU

Bring U onto A_0

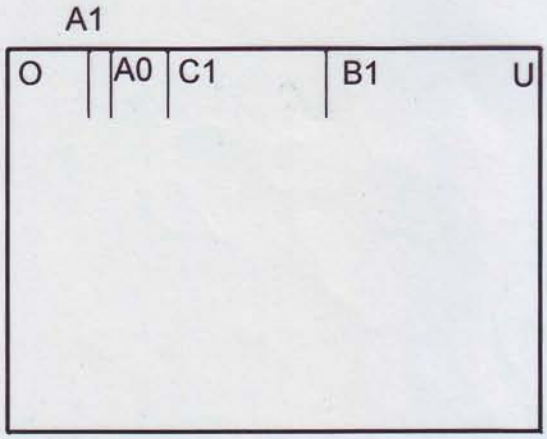
to determine B_1

L



O onto B_1 to determine C_1

R



O onto C_1 to determine A_1

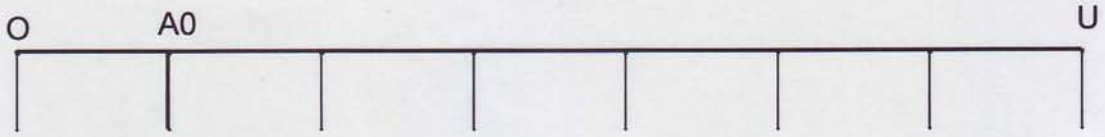
R

Repeat starting from A_1, etc.

When $A_{m+1} \approx A_m$, $OA_m \approx \frac{1}{7} OU$.

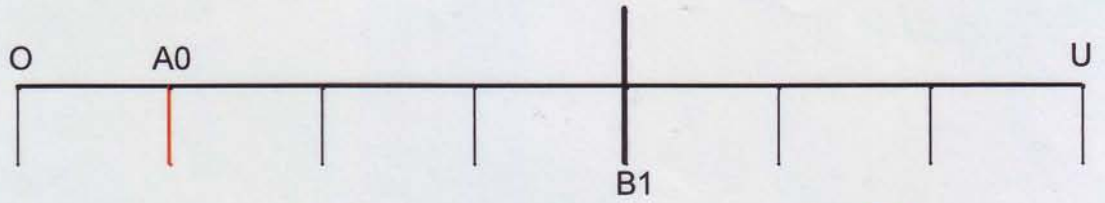
Why is it working?

Set $OU = 1$. Then



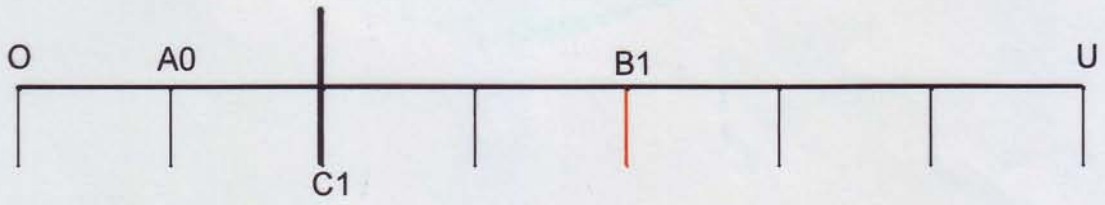
$OA_0 = \frac{1}{7} + E$ L

$A_0U = \frac{6}{7} - E$

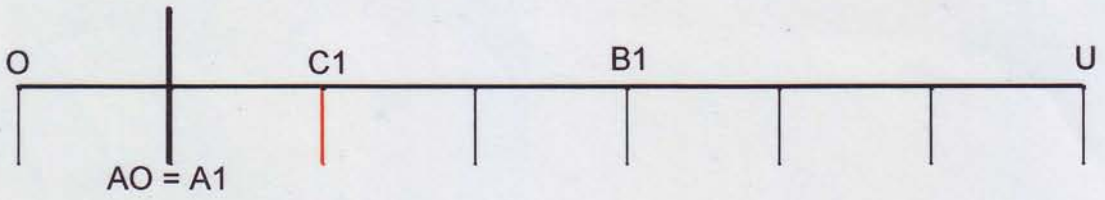


$B_1U = \frac{3}{7} - \frac{E}{2}$

$OB_1 = \frac{4}{7} + \frac{E}{2}$ R



$OC_1 = \frac{2}{7} + \frac{E}{4}$ R



$OA_1 = \frac{1}{7} + \frac{E}{8}$

...

$OA_m = \frac{1}{7} + \frac{E}{8^m}$

8^n increases very quickly $\Rightarrow$

$\frac{E}{8^n}$ decreases very quickly

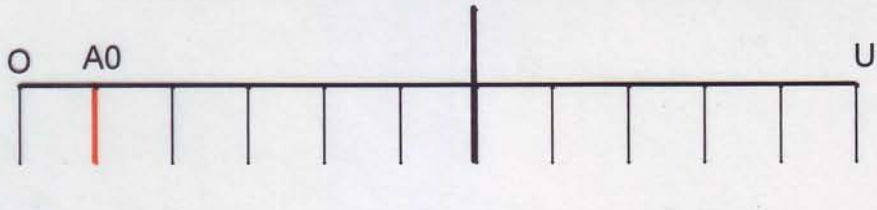
If we choose A_0 "conveniently", we are done in 2-3 iterations

The "recipe" for folding $\frac{1}{7}$ via Fujimoto approximation is LRR (RLL for some authors).

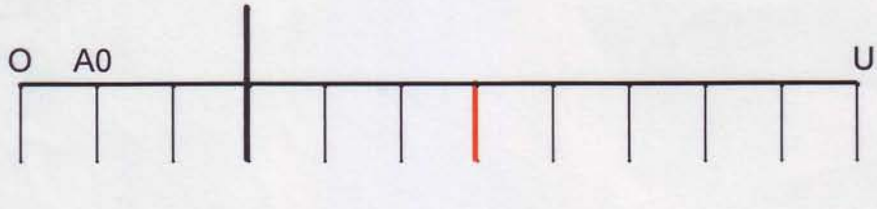
It is made of 3 letters and at each iteration the error is divided by 2^3.

How can we find the recipe for any odd number?

DIVISION IN 11 EQUAL PARTS



L



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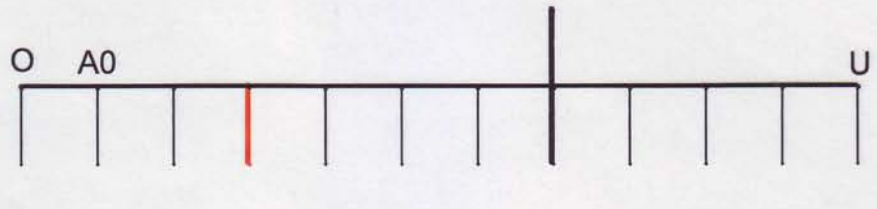

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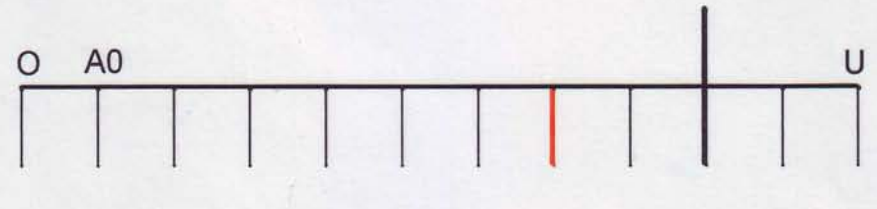

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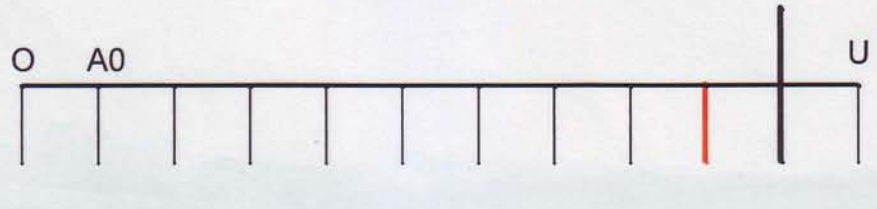

L

The recipe for folding $\frac{1}{11}$ is made of $5+5=10$ letters $\Rightarrow$ after just one iteration, the error is divided by $2^{10}=1024$.
If the sheet is not too big, one iteration suffices!

In general, the longer the recipe, the fewer times we have to iterate it.

For instance, for $\frac{1}{53}$ the recipe is

$$\underbrace{LLRRR\ldots LLLLL}_{26}\ \underbrace{RRLLL\ldots RRRRR}_{26}$$

$$\Rightarrow E \rightsquigarrow \frac{E}{2^{26}} = \frac{E}{67\,108\,864}$$

$$\rightsquigarrow \frac{E}{2^{52}} = \frac{E}{4\,503\,599\,627\,370\,496}!$$

Final remark The recipe for Fujimoto's approximation is strictly related with the representation of $\frac{1}{N}$ in base 2.

$\frac{1}{7}$ $\quad \frac{1}{111} = 0,\overline{001}$ LRR

$\frac{1}{11}$ $\quad \frac{1}{1011} = 0,\overline{0001011101}$ LRLLLRLRRR

For more details on exact subdivisions, see

Emma Frigerio
Variations on a Theorem of Haga
in Origami^5: Fifth International Meeting of Origami Science, Mathematics, and Education, edited by Patsy Wang-Iverson, Robert J. Lang, Mark Yim, pp. 507-517, 2011